7.1 Diagonalization of Symmetric Matrices

A **symmetric matrix** is a matrix A such that $A^T = A$.

For example,

$$\begin{bmatrix}1 & 0\\0 & -3\end{bmatrix}, \quad \begin{bmatrix}0 & -1 & 0\\-1 & 5 & 8\\0 & 8 & -7\end{bmatrix}, \quad \begin{bmatrix}a & b & c\\b & d & e\\c & e & f\end{bmatrix} \text{ are symmetric.}$$

$$\begin{bmatrix}1 & -3\\3 & 0\end{bmatrix}, \begin{bmatrix}1 & -4 & 0\\-6 & 1 & -4\\0 & -6 & 1\end{bmatrix}, \quad \begin{bmatrix}5 & 4 & 3 & 2\\4 & 3 & 2 & 1\\3 & 2 & 1 & 0\end{bmatrix} \text{ are nonsymmetric.}$$

An **orthogonal matrix** is a real square matrix whose columns and rows are orthonormal vectors. Equivalently, a matrix P is orthogonal if its transpose is equal to its inverse: $P^T = P^{-1}$. $\Longleftrightarrow$ $P^T P = I$

Example 1. Determine which of the matrices below are orthogonal. If orthogonal, find the inverse.

(1) $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

(2) $\begin{bmatrix}1/3 & 2/3 & 2/3\\2/3 & 1/3 & -2/3\\2/3 & -2/3 & 1/3\end{bmatrix}$

ANS: (1) Note P is square. $P^T P = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = 2I \neq I$.

P is not orthogonal.

(2) P is a square matrix.

$$P^T P = \begin{bmatrix}1/3 & 2/3 & 2/3\\2/3 & 1/3 & -2/3\\2/3 & -2/3 & 1/3\end{bmatrix}\begin{bmatrix}1/3 & 2/3 & 2/3\\2/3 & 1/3 & -2/3\\2/3 & -2/3 & 1/3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = I.$$

Thus P is orthogonal and $P^{-1} = P^T = \begin{bmatrix}1/3 & 2/3 & 2/3\\2/3 & 1/3 & -2/3\\2/3 & -2/3 & 1/3\end{bmatrix}$

Notice A is symmetric.

Example 2. Diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

ANS: Recall to diagonalize A, we need to find an invertible P and diagonal D such $A = PDP^{-1}$.

Exercise: Check the eigenvalues and eigenvectors of A are

$$\lambda_1 = 8, \ \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 6, \ \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad \lambda_3 = 3, \ \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Notice that, $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$ and $\vec{v}_1 \cdot \vec{v}_3 = 0$, i.e. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

We can normalize $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to get an orthonormal basis.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \quad \vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Let $P = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$. $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Then $A = PDP^{-1} = PDP^T$

Note P is an orthogonal matrix (P is square and has orthonormal columns), thus $P^{-1} = P^T$

The next theorem explains why the eigenvectors for A are orthogonal (Since A is symmetric and they come from distinct evs)

Theorem 1. If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

- An $n \times n$ matrix A is said to be orthogonally diagonalizable if there are an orthogonal matrix P (with $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1} \tag{1}$$

- If A is orthogonally diagonalizable as in (1), then

$$\underline{A^T} = (PDP^T)^T = P^{TT}D^TP^T = PDP^T = \underline{A}$$

- Thus A is symmetric. Conversely, every symmetric matrix is orthogonally diagonalizable as in Theorem 2:

Theorem 2. An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

- In particular, a symmetric matrix is always diagonalizable.

Example 3. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 5 is an eigenvalue of A and $\mathbf{v}$ is an eigenvector. Then orthogonally diagonalize A.

ANS: We can either follow the standard calculation to find the eigenvalues and eigenvectors, or we use the given information:

$$\underline{A\vec{v}} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underline{2\vec{v}}$$

Thus the given $\vec{v}$ is an eigenvector corresponds to $\lambda = 2$.

To verify 5 is an eigenvalue, we solve $(A-5I)\vec{v} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This means $(A-5I)\vec{v} = \vec{0}$ has nontrivial solutions, so 5 is an eigenvalue for A. (Since if 5 is not an eigenvalue, $A\vec{v} = 5\vec{v}$ if and only if $\vec{v} = \vec{0}$ by the def of eigenvalue).

Moreover.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$ form a basis for

the eigenspace corresponding to $\lambda = 5$.

Since $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal, we can use the Gram-Schmidt process to find the orthonormal basis.

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

We update $\vec{u}_1$ and $\vec{u}_2$ by normalizing them,

$$\vec{u}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

We also normalize $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to get $\vec{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

Let $P = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Then P orthogonally diagonalizes A and

$$A = PDP^{-1} = PDP^{T}$$

The Spectral Theorem

The set of eigenvalues of a matrix A is sometimes called the **spectrum** of A, and the following description of the eigenvalues is called a **spectral theorem**.

Theorem 3. The Spectral Theorem for Symmetric Matrices
An $n \times n$ symmetric matrix A has the following properties:
a. A has n real eigenvalues, counting multiplicities.
b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
d. A is orthogonally diagonalizable.

Spectral Decomposition

Suppose $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of A and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are in the diagonal matrix D. Then, since $P^{-1} = P^T$,

$$\begin{aligned} A = PDP^T &= [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \\ &= [\lambda_1 \mathbf{u}_1 \quad \cdots \quad \lambda_n \mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \end{aligned}$$

Using the column-row expansion of a product (Theorem 10 in Section 2.4), we can write

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \tag{2}$$

- This representation of A is called a **spectral decomposition** of A because it breaks up A into pieces determined by the spectrum (eigenvalues) of A.

Example 4. Construct a spectral decomposition of A from Example 2.

ANS: Recall in Example 2.

$$P = [\vec{u}_1 \quad \vec{u}_2 \quad u_3] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \qquad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then $$A = 8\vec{u}_1\vec{u}_1^T + 6\,\vec{u}_2\vec{u}_2^T + 3\,u_3\,\vec{u}_3^T$$

Exercise: Verify the above equation holds.

Answer:

$$8\vec{u}_1\vec{u}_1^T = 8\cdot\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix} = \begin{bmatrix}4 & -4 & 0\\ -4 & 4 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$6\vec{u}_2\vec{u}_2^T = 6\begin{bmatrix}-\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{bmatrix} = \begin{bmatrix}1 & 1 & -2\\ 1 & 1 & -2\\ -2 & -2 & 4\end{bmatrix}$$

$$3\vec{u}_3\vec{u}_3^T = 3\begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Thus

$$8\vec{u}_1\vec{u}_1^T + 6\,\vec{u}_2\vec{u}_2^T + 3\,u_3\,\vec{u}_3^T = \begin{bmatrix}6 & -2 & -1\\ -2 & 6 & -1\\ -1 & -1 & 5\end{bmatrix} = A$$

Exercise 5. Suppose $A = PRP^{-1}$, where P is orthogonal and R is lower triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

Solution. If $A = PRP^{-1}$, then $P^{-1}AP = R$. Since P is orthogonal, $R = P^TAP$. Hence $R^T = \left(P^TAP\right)^T = P^TA^TP^{TT} = P^TAP = R$, which shows that R is symmetric. Since R is also lower triangular, its entries below the diagonal must be zeros to match the zeros above the diagonal. Thus R is a diagonal matrix.

Exercise 6. Orthogonally diagonalize the matrices given below, giving an orthogonal matrix P and a diagonal matrix D.

(1) $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$

(3) $\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$

Solution.

(1) Let $A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$. Then the characteristic polynomial of A is $(1-\lambda)^2 - 25 = \lambda^2 - 2\lambda - 24$ $= (\lambda - 6)(\lambda + 4)$, so the eigenvalues of A are 6 and -4. For $\lambda = 6$, one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. For $\lambda = -4$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$

(2) Let $A=\begin{bmatrix}1 & -6 & 4\\ -6 & 2 & -2\\ 4 & -2 & -3\end{bmatrix}$. The eigenvalues of A are $-3, -6$ and 9 . For $\lambda=-3$, one computes that a basis for the eigenspace is $\begin{bmatrix}1\\2\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_1=\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$. For $\lambda=-6$, one computes that a basis for the eigenspace is $\begin{bmatrix}-2\\-1\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_2=\begin{bmatrix}-2/3\\-1/3\\2/3\end{bmatrix}$. For $\lambda=9$, one computes that a basis for the eigenspace is $\begin{bmatrix}2\\-2\\1\end{bmatrix}$, which can be normalized to get $\mathbf{u}_3=\begin{bmatrix}2/3\\-2/3\\1/3\end{bmatrix}$. Let $P=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]=\begin{bmatrix}1/3 & -2/3 & 2/3\\ 2/3 & -1/3 & -2/3\\ 2/3 & 2/3 & 1/3\end{bmatrix}$ and $D=\begin{bmatrix}-3 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & 9\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A=PDP^{-1}$.

(3) Let $A=\begin{bmatrix}5 & 8 & -4\\ 8 & 5 & -4\\ -4 & -4 & -1\end{bmatrix}$. The eigenvalues of A are -3 and 15. For $\lambda=-3$, one computes that a which is orthogonal and can be normalized to get basis for the eigenspace is $\left\{\begin{bmatrix}2\\-1\\2\end{bmatrix},\begin{bmatrix}-1\\2\\2\end{bmatrix}\right\}$ $\{\mathbf{u}_1,\mathbf{u}_2\}=\left\{\begin{bmatrix}2/3\\-1/3\\2/3\end{bmatrix},\begin{bmatrix}-1/3\\2/3\\2/3\end{bmatrix}\right\}$. For $\lambda=15$, one computes that a basis for the eigenspace is $\begin{bmatrix}2\\2\\-1\end{bmatrix}$ $D=\begin{bmatrix}-3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 15\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A=PDP^{-1}$.

Exercise 7. Suppose A and B are both orthogonally diagonalizable and $AB=BA$. Explain why AB is also orthogonally diagonalizable.

Solution. If A and B are orthogonally diagonalizable, then A and B are symmetric by Theorem 2 . If $AB=BA$, then $(AB)^T=(BA)^T=A^TB^T=AB$. So AB is symmetric and hence is orthogonally diagonalizable by Theorem 2 .